

# COMPACT FUZZY SOFT SPACES

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**ABSTRACT.** In this article, by using basic properties of fuzzy soft topology we defined fuzzy soft compactness. We also introduced some basic definitions and theorems of the concept.

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## 1. INTRODUCTION

There are many problems encountered in real life. Various mathematical set theories such as soft set which was introduced by Molodtsov [7] and fuzzy set which developed by Zadeh [10] have been developed to solve these problems. P. K. Maji, R. Biswas, A. R. Roy [11] also initiated the more generalized concept of fuzzy soft sets which is a combination of fuzzy set and soft set. Then many researchers have applied this concept. B. Tanay et. al. introduced topological structure of fuzzy soft set in [5] and gave a introductory theoretical base to carry further study on this concept. Following this study, some others ([6],[9],[12],[13]) studied on the concept of fuzzy soft topological spaces. We will introduce compactness on fuzzy soft topological spaces and give some important definitions and theorems.

## 2. PRELIMINARIES

**Definition 2.1.** [10] *A fuzzy set  $A$  of a non-empty set  $X$  is characterized by a membership function  $\mu_A : X \rightarrow [0, 1]$  whose value  $\mu_A(x)$  represents the "grade of membership" of  $x$  in  $A$  for  $x \in X$ .*

*Let  $I^X$  denotes the family of all fuzzy sets on  $X$ . If  $A, B \in I^X$ , then some basic set operations for fuzzy sets are given by Zadeh as follows:*

- (1):  $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ , for all  $x \in X$ .
- (2):  $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ , for all  $x \in X$ .
- (3):  $C = A \vee B \Leftrightarrow \mu_C(x) = \mu_A(x) \vee \mu_B(x)$ , for all  $x \in X$ .
- (4):  $D = A \wedge B \Leftrightarrow \mu_D(x) = \mu_A(x) \wedge \mu_B(x)$ , for all  $x \in X$ .
- (5):  $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$ , for all  $x \in X$ .

**Definition 2.2.** [7] *Let  $X$  be the initial universe set and  $E$  be the set of parameters. A pair  $(F, A)$  is called a soft set over  $X$  where  $F$  is a mapping given by  $F : A \rightarrow P(X)$  and  $A \subseteq E$ .*

*In the other words, the soft set is a parametrized family of subsets of the set  $X$ . Every set  $F(e)$ , for every  $e \in A$ , from this family may be considered as the set of  $e$ -elements of the soft set  $(F, A)$ .*

**Definition 2.3.** [11] Let  $A \subseteq E$ . A pair  $(f, A)$  is called a fuzzy soft set over  $X$ , where  $f : A \rightarrow I^X$  is a function.

That is, for each  $a \in A$ ,  $f(a) = f_a : X \rightarrow I$  is a fuzzy set on  $X$ .

**Definition 2.4.** [6] Fuzzy soft set  $(f, A)$  on the universe  $X$  is a mapping from the parameter set  $E$  to  $I^X$ , i.e.,  $(f, A) : E \rightarrow I^X$ , where  $(f, A)(e) \neq 0_X$  if  $e \in A \subseteq E$  and  $(f, A)(e) = 0_X$  if  $e \notin A$ , where  $0_X$  is empty fuzzy set on  $X$ .

From now on, we will use  $FS(X, E)$  instead of the family of all fuzzy soft sets over  $X$ .

**Definition 2.5.** [6] Let  $(f, A), (g, B) \in FS(X, E)$ . The following operations are defined as follows:

**Subset:**  $(f, A) \tilde{\subseteq} (g, B)$  if  $(f, A)(e) \leq (g, B)(e)$ , for each  $e \in E$ .

**Equal:**  $(f, A) = (g, B)$  if  $(f, A) \tilde{\subseteq} (g, B)$  and  $(g, B) \tilde{\subseteq} (f, A)$ .

**Union:**  $(h, A \cup B) = (f, A) \tilde{\cup} (g, B)$  where  $(h, A \cup B)(e) = (f, A)(e) \vee (g, B)(e)$ , for all  $e \in E$ .

**Intersection:**  $(h, A \cap B) = (f, A) \tilde{\cap} (g, B)$  where  $(h, A \cap B)(e) = (f, A)(e) \wedge (g, B)(e)$ , for all  $e \in E$ .

**Definition 2.6.** [6] Let  $(f, A) \in FS(X, E)$ . Then complement of  $(f, A)$ , denoted by  $(f, A)^c$ , is the fuzzy soft set defined by  $(f, A)^c(e) = 1_X - (f, A)(e)$ , for all  $e \in E$ .

Clearly  $((f, A)^c)^c = (f, A)$ .

**Definition 2.7.** [6] Let  $(f, E) \in FS(X, E)$ . The fuzzy soft set  $(f, E)$  is called the null fuzzy soft set, denoted by  $\tilde{0}_E$ , if  $(f, E)(e) = 0_X$ , for all  $e \in E$ .

**Definition 2.8.** [6] Let  $(f, E) \in FS(X, E)$ . The fuzzy soft set  $(f, E)$  is called the universal fuzzy soft set, denoted by  $\tilde{1}_E$ , if  $(f, E)(e) = 1_X$ , for all  $e \in E$ .

Clearly  $(\tilde{1}_E)^c = \tilde{0}_E$  and  $(\tilde{0}_E)^c = \tilde{1}_E$ .

**Definition 2.9.** [2] Let  $FS(X, E)$  and  $FS(Y, K)$  be the families of all fuzzy soft sets over  $X$  and  $Y$ , respectively. Let  $\varphi : X \rightarrow Y$  and  $\psi : E \rightarrow K$  be two functions. Then the pair  $(\varphi, \psi)$  is called a fuzzy soft mapping from  $X$  to  $Y$ , and denoted by  $(\varphi, \psi) : FS(X, E) \rightarrow FS(Y, K)$ .

If  $\varphi$  and  $\psi$  is injective then the fuzzy soft mapping  $(\varphi, \psi)$  is said to be injective.

If  $\varphi$  and  $\psi$  is surjective then the fuzzy soft mapping  $(\varphi, \psi)$  is said to be surjective.

The fuzzy soft mapping  $(\varphi, \psi)$  is called constant, if  $\varphi$  and  $\psi$  are constant.

**Definition 2.10.** [5] A fuzzy soft topological space is a pair  $(X, \tau)$  where  $X$  is a nonempty set and  $\tau$  a family of fuzzy soft sets over  $X$  satisfying the following properties:

- (1)  $\tilde{0}_E, \tilde{1}_E \in \tau$ ,
- (2) If  $(f, A), (g, B) \in \tau$ , then  $(f, A) \tilde{\cap} (g, B) \in \tau$
- (3) If  $(f_i, A) \in \tau, i \in J$ , then  $\cup_{i \in J} (f_i, A) \in \tau$

$\tau$  is called a topology of fuzzy soft sets on  $X$ . Every member of  $\tau$  is called fuzzy soft open.

$(g, B)$  is called fuzzy soft closed in  $(X, \tau)$  if  $(g, B)^c \in \tau$ .

**Definition 2.11.** [5] Let  $(X, \tau_1)$  and  $(Y, \tau_2)$  be two fuzzy soft topological spaces. If each  $(f, A) \in \tau_1$  is in  $\tau_2$ , then  $\tau_2$  is called fuzzy soft finer than  $\tau_1$ , or (equivalently)  $\tau_1$  is fuzzy soft coarser than  $\tau_2$ .

**Definition 2.12.** [6] Let  $(X, \tau_1)$  and  $(Y, \tau_2)$  be two fuzzy soft topological spaces.

(1) A fuzzy soft mapping  $(\varphi, \psi) : (X, \tau_1) \rightarrow (Y, \tau_2)$  is called fuzzy soft continuous if  $(\varphi, \psi)^{-1}((g, B)) \in \tau_1, \forall (g, B) \in \tau_2$ .

(2) A fuzzy soft mapping  $(\varphi, \psi) : (X, \tau_1) \rightarrow (Y, \tau_2)$  is called fuzzy soft open if  $(\varphi, \psi)((f, A)) \in \tau_2, \forall (f, A) \in \tau_1$ .

### 3. COMPACT FUZZY SOFT SPACES

**Definition 3.1.** A family  $\Psi$  of fuzzy soft sets is a cover of a fuzzy soft set  $(f, A)$  if  $(f, A) \subseteq \cup \{(f_i, A) : (f_i, A) \in \Psi, i \in I\}$ .

It is a fuzzy soft open cover if each member of  $\Psi$  is a fuzzy soft open set. A subcover of  $\Psi$  is a subfamily of  $\Psi$  which is also a cover.

**Definition 3.2.** Let  $(X, \tau)$  be fuzzy soft topological space and  $(f, A) \in FS(X, E)$ . Fuzzy soft set  $(f, A)$  is called compact if each fuzzy soft open cover of  $(f, A)$  has a finite subcover. Also fuzzy soft topological space  $(X, \tau)$  is called compact if each fuzzy soft open cover of  $\tilde{1}_E$  has a finite subcover.

**Example 3.3.** A fuzzy soft topological space  $(X, \tau)$  is compact if  $X$  is finite.

**Example 3.4.** Let  $(X, \tau)$  and  $(Y, \sigma)$  be two fuzzy soft topological spaces and  $\tau \subset \sigma$ . Then, fuzzy soft topological space  $(X, \tau)$  is compact if  $(Y, \sigma)$  is compact.

**Proposition 3.5.** Let  $(g, B)$  be a fuzzy soft closed set in fuzzy soft compact space  $(X, \tau)$ . Then  $(g, B)$  is also compact.

*Proof.* Let  $(f_i, A)$  be any open covering of  $(g, B)$ . Then  $\tilde{1}_X \subseteq (\cup_{i \in I} (f_i, A)) \cup (g, B)^c$ , that is,  $(f_i, A)$  together with fuzzy soft open set  $(g, B)^c$  is a open covering of  $\tilde{1}_X$ . Therefore there exists a finite subcovering  $(f_1, A), (f_2, A), \dots, (f_n, A), (g, B)^c$ . Hence we obtain  $\tilde{1}_X \subseteq (f_1, A) \cup (f_2, A) \cup \dots \cup (f_n, A) \cup (g, B)^c$ . Therefore, we get  $(g, B) \subseteq (f_1, A) \cup (f_2, A) \cup \dots \cup (f_n, A) \cup (g, B)^c$  which clearly implies  $(g, B) \subseteq (f_1, A) \cup (f_2, A) \cup \dots \cup (f_n, A)$  since  $(g, B) \cap (g, B)^c = \Phi$ . Hence  $(g, B)$  has a finite subcovering and so is compact.  $\square$

**Definition 3.6.** [9] Let  $(X, \tau)$  be a fuzzy soft topological space over  $X$  and  $x, y \in X$  such that  $x \neq y$ . If there exist fuzzy soft open sets  $(f, A)$  and  $(g, A)$  such that  $x \in (f, A), y \in (g, A)$  and  $(f, A) \tilde{\cap} (g, A) = \Phi$ , then  $(X, \tau)$  is called a fuzzy soft Hausdorff space.

**Proposition 3.7.** Let  $(g, B)$  be a fuzzy soft compact set in fuzzy soft Hausdorff space  $(X, \tau)$ . Then  $(g, B)$  is closed.

*Proof.* Let  $x \in (g, B)^c$ . For each  $y \in (g, B)$ , we have  $x \neq y$ , so there are disjoint fuzzy soft open sets  $(f_y, A)$  and  $(h_y, A)$  so that  $x \in (f_y, A)$  and  $y \in (h_y, A)$ . Then  $\{(h_y, A) : y \in (g, B)\}$  is an fuzzy soft open cover of  $(g, B)$ . Let  $\{(h_{y_1}, A), (h_{y_2}, A), \dots, (h_{y_n}, A)\}$  be a finite subcover. Then  $\cap_{i=1}^n (f_{y_i}, A)$  is an open set containing  $x$  and contained in  $(g, B)^c$ . Thus  $(g, B)^c$  is fuzzy soft open and  $(g, B)$  is closed.  $\square$

**Theorem 3.8.** Let  $(X, \tau)$  and  $(Y, \sigma)$  be fuzzy soft topological spaces and  $(\varphi, \psi) : (X, \tau) \rightarrow (Y, \sigma)$  continuous and onto fuzzy soft function. If  $(X, \tau)$  is fuzzy soft compact, then  $(Y, \sigma)$  is fuzzy soft compact,

*Proof.* We will use Theorem 3.8. and Theorem 3.10. of [3]. Let  $(f_i, A)$  be any open covering of  $\tilde{1}_Y$ , i.e.,  $\tilde{1}_Y \subseteq \cup_{i \in I} (f_i, A)$ . Then  $(\varphi, \psi)^{-1}(\tilde{1}_Y) \subseteq (\varphi, \psi)^{-1}(\cup_{i \in I} (f_i, A))$  and  $\tilde{1}_X \subseteq \cup_{i \in I} (\varphi, \psi)^{-1}((f_i, A))$ . So  $(\varphi, \psi)^{-1}((f_i, A))$  is an open covering of  $\tilde{1}_X$ . As  $(X, \tau)$  is compact, there are  $1, 2, \dots, n$  in  $I$  such that

$$\tilde{1}_X \subseteq (\varphi, \psi)^{-1}((f_1, A)) \cup (\varphi, \psi)^{-1}((f_2, A)) \cup \dots \cup (\varphi, \psi)^{-1}((f_n, A)).$$

Since  $(\varphi, \psi)$  is surjective, we have

$$\begin{aligned} \tilde{1}_Y &= (\varphi, \psi)(\tilde{1}_X) \\ &\subseteq (\varphi, \psi)\left((\varphi, \psi)^{-1}((f_1, A)) \cup \dots \cup (\varphi, \psi)^{-1}((f_n, A))\right) \\ &= (\varphi, \psi)\left((\varphi, \psi)^{-1}((f_1, A))\right) \cup \dots \cup (\varphi, \psi)\left((\varphi, \psi)^{-1}((f_n, A))\right) \\ &= (f_1, A) \cup (f_2, A) \cup \dots \cup (f_n, A). \end{aligned}$$

So we have  $\tilde{1}_Y \subseteq (f_1, A) \cup (f_2, A) \cup \dots \cup (f_n, A)$ , i.e.,  $\tilde{1}_Y$  is covered by a finite number of  $(f_i, A)$ .

Hence  $(Y, \sigma)$  is compact.  $\square$

**Definition 3.9.** Let  $(X, \tau)$  and  $(Y, \sigma)$  be two fuzzy soft topological spaces. A fuzzy soft mapping  $(\varphi, \psi) : (X, \tau) \rightarrow (Y, \sigma)$  is called fuzzy soft closed if  $(\varphi, \psi)((f, A))$  is fuzzy soft closed set in  $(Y, \sigma)$ , for all fuzzy soft closed set  $(f, A)$  in  $(X, \tau)$ .

**Theorem 3.10.** Let  $(X, \tau)$  be a fuzzy soft topological space and  $(Y, \sigma)$  be a fuzzy soft Hausdorff space. Fuzzy soft mapping  $(\varphi, \psi)$  is closed if fuzzy soft mapping  $(\varphi, \psi) : (X, \tau) \rightarrow (Y, \sigma)$  is continuous.

*Proof.* Let  $(g, B)$  be any fuzzy soft closed set in  $(X, \tau)$ . By theorem 3.5 we have  $(g, B)$  is compact. Since fuzzy soft mapping  $(\varphi, \psi)$  is continuous, fuzzy soft set  $(\varphi, \psi)((g, B))$  is compact in  $(Y, \sigma)$ . As  $(Y, \sigma)$  is fuzzy soft Hausdorff space, fuzzy soft set  $(\varphi, \psi)((g, B))$  is closed. Then Fuzzy soft mapping  $(\varphi, \psi)$  is closed.  $\square$

**Definition 3.11.** A family  $\Psi$  of fuzzy soft sets has the finite intersection property if the intersection of the members of each finite subfamily of  $\Psi$  is not the null fuzzy soft set.

**Theorem 3.12.** A fuzzy soft topological space is compact if and only if each family of fuzzy soft closed sets with the finite intersection property has a nonnull intersection.

*Proof.*  $\Rightarrow$ : Let  $\Psi$  be any family of fuzzy soft closed subset such that  $\cap\{(f_i, A) : (f_i, A) \in \Psi, i \in I\} = \tilde{0}_E$ . Consider  $\Omega = \{(f_i, A)^c : (f_i, A) \in \Psi, i \in I\}$ . So  $\Omega$  is a fuzzy soft open cover of  $\tilde{1}_E$ . As fuzzy soft topological space is compact, there exists a finite subcovering  $(f_1, A)^c, (f_2, A)^c, \dots, (f_n, A)^c$ . Then  $\cap_{i=1}^n (f_i, A) = \tilde{1}_E - \cup_{i=1}^n (f_i, A)^c = \tilde{1}_E - \tilde{1}_E = \tilde{0}_E$ . Hence  $\Psi$  can not have finite intersection property.

$\Leftarrow$ : Assume that a fuzzy soft topological space is not compact. Then any fuzzy soft open cover of  $\tilde{1}_E$  has not a finite subcover. Let  $\{(f_i, A) : i \in I\}$  be fuzzy soft open cover of  $\tilde{1}_E$ . So  $\cup_{i=1}^n (f_i, A) \neq \tilde{1}_E$ . Therefore  $\cap_{i=1}^n (f_i, A)^c \neq \tilde{0}_E$ . Thus,  $\{(f_i, A)^c : i = 1, \dots, n\}$  have finite intersection property. By using hypothesis,  $\cap (f_i, A)^c \neq \tilde{0}_E$  and we have  $\cup (f_i, A) \neq \tilde{1}_E$ . This is a contradiction. Thus the fuzzy soft topological space is compact.  $\square$

## 4. CONCLUSION

In this work, we introduced fuzzy soft compactness and gave basic definitions and theorems of this concept. Also we introduced fuzzy soft cover, fuzzy soft subcover, fuzzy soft open cover.

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